FSTTCS 2023

Weighted One-Deterministic-Counter Automata

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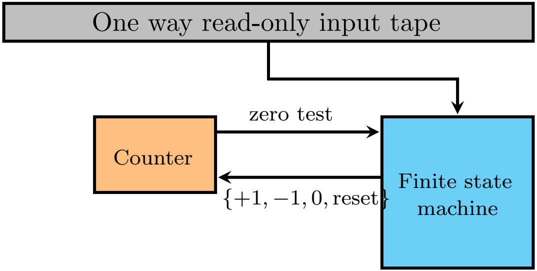
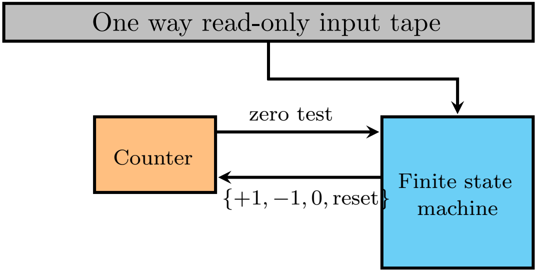
Outline of the talk

1. One-counter Automata
2. One-deterministic-counter automata (ODCA)
   - Motivation
   - Our equivalence results
   - Reachability problem of ODCA
   - A quick look at other results

One-counter automata (OCA)

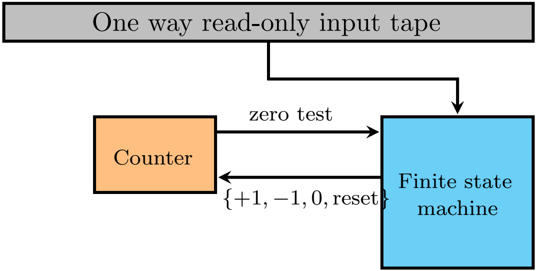


One-counter automata (OCA)



Write to counter: Increment(+1), No change (0), Decrement (-1)

One-counter automata (OCA)



Write to counter: Increment(+1), No change (0), Decrement (-1)
Read from counter: zero (0), or positive (+)

Results - OCA

- Equivalence of deterministic OCA is in P [2].
- Equivalence of non-deterministic OCA is undecidable [7].
- Equivalence of weighted OCA is open (weights from a field).

One-deterministic-counter automata (ODCA)

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Semantic definition

An OCA where all runs of a word lead to the same counter value.

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An OCA where all runs of a word lead to the same counter value.

Syntactic definition

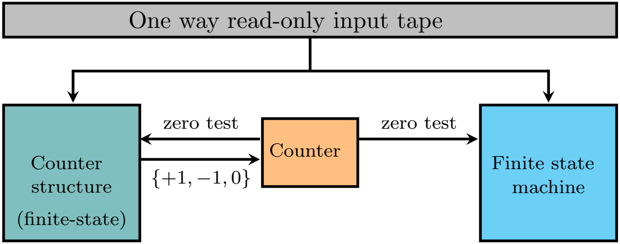


Figure: One-deterministic-counter automata

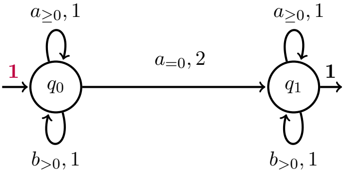
Example - Weighted ODCA (visibly)

Example

$$\texttt{eqPrefix}(w) = \begin{cases} \mathbf{2} \cdot \mathbf{k}, \text{where } k \text{ is the \#proper prefixes of } w \text{ with} \\ \qquad \#a\text{'s} = \ \#b\text{'s, if } \#a\text{'s} \geq \#b\text{'s for all prefixes} \\ \mathbf{0}, \ \text{otherwise} \end{cases}$$

.

- ODCAs are strict extensions of visibly OCA.



Finite state machine

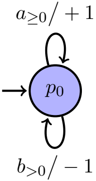
a	a	b	b	a	b	a

Input tape

0

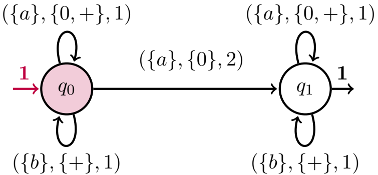
Counter

Counter structure



1	0

Initial vector



Finite state machine

a	a	b	b	a	b	a

Input tape

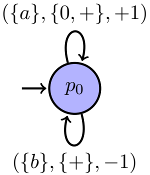
0

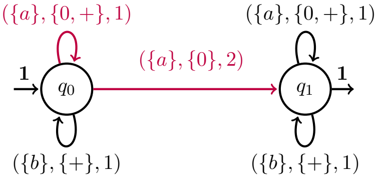
Counter

1	0

Initial vector

Counter structure





Finite state machine

a	a	b	b	a	b	a

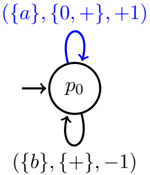
Input tape

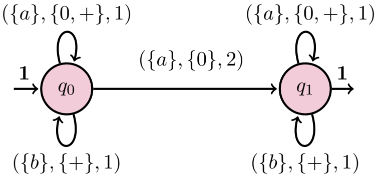
0

Counter

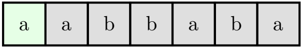
1	0

Counter structure



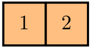


Finite state machine

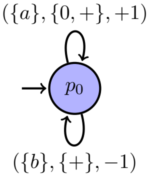


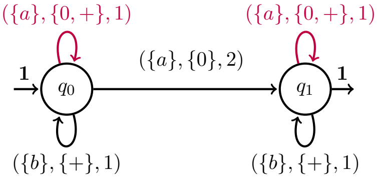
Input tape

Counter



Counter structure





Finite state machine

a	a	b	b	a	b	a

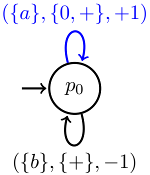
Input tape

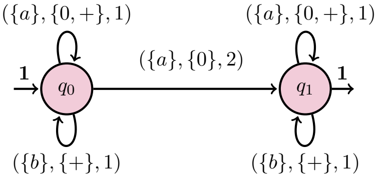
1

Counter

1	2

Counter structure





Finite state machine

a	a	b	b	a	b	a

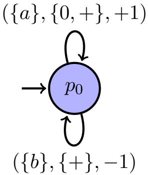
Input tape

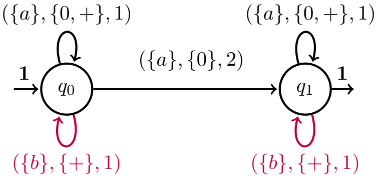
2

Counter

1	2

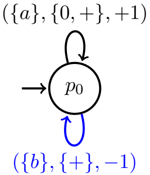
Counter structure





Finite state machine

Counter structure



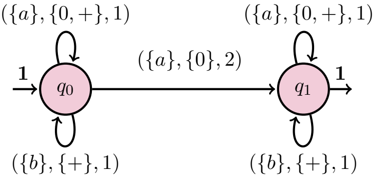
a	a	b	b	a	b	a

Input tape

2

Counter

1	2



Finite state machine

a	a	b	b	a	b	a

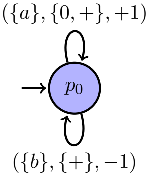
Input tape

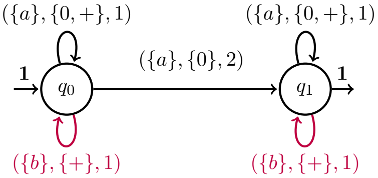
1

Counter

1	2

Counter structure





Finite state machine

a	a	b	b	a	b	a

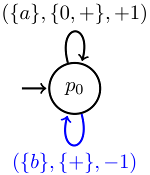
Input tape

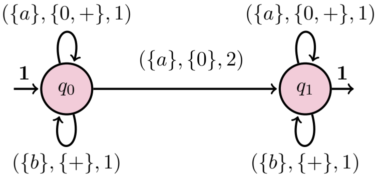
1

Counter

1	2

Counter structure





Finite state machine

a	a	b	b	a	b	a

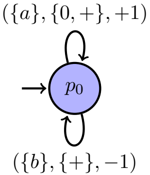
Input tape

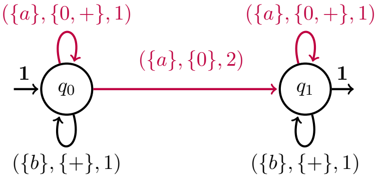
0

Counter

1	2

Counter structure





Finite state machine

a	a	b	b	a	b	a

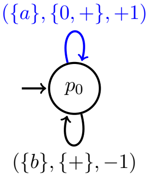
Input tape

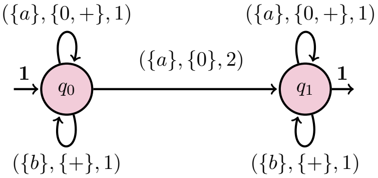
0

Counter

1	2

Counter structure





Finite state machine

a	a	b	b	a	b	a

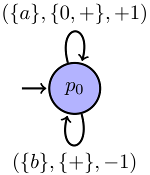
Input tape

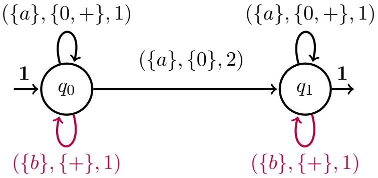
1

Counter

1	4

Counter structure





Finite state machine

a	a	b	b	a	b	a

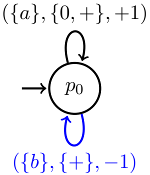
Input tape

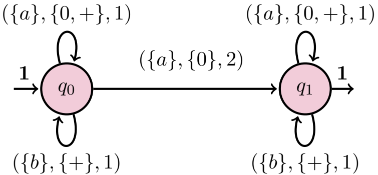
1

Counter

1	4

Counter structure





Finite state machine

a	a	b	b	a	b	a

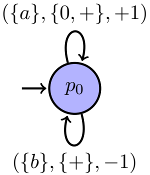
Input tape

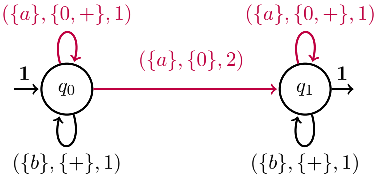
0

Counter

1	4

Counter structure





Finite state machine

a	a	b	b	a	b	a

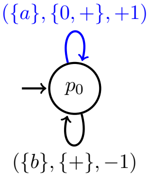
Input tape

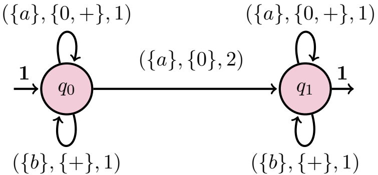
0

Counter

1	4

Counter structure





Finite state machine

a	a	b	b	a	b	a

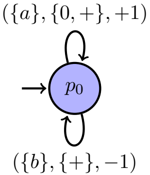
Input tape

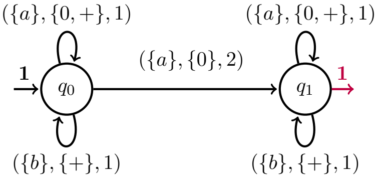
1

Counter

1	6

Counter structure





Finite state machine

a	a	b	b	a	b	a

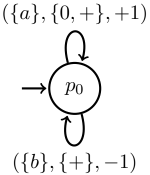
Input tape

1

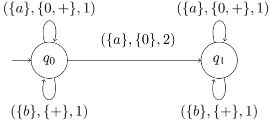
Counter

1	6

Counter structure



Weighted visibly ODCA- Evaluation



Initial distribution

λ

1	0

Transition matrices

$\delta_0(a)$

1	2
0	1

$\delta_+(a)$

1	0
0	1

$\delta_+(b)$

1	0
0	1

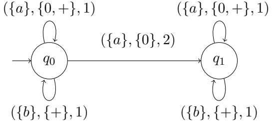
Final distribution

ρ

0
1

- Let $w = aababba$.

Weighted visibly ODCA- Evaluation



Initial distribution | Transition matrices | Final distribution

λ

1	0

$\delta_0(a)$

1	2
0	1

$\delta_+(a)$

1	0
0	1

$\delta_+(b)$

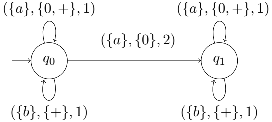
1	0
0	1

ρ

0
1

- Let $w = aababba$.
- $\delta(w) = \delta_0(a)\ \delta_+(a)\ \delta_+(b)\ \delta_+(a)\ \delta_+(b)\ \delta_+(b)\ \delta_0(a)$.

Weighted visibly ODCA- Evaluation



Initial distribution

λ

1	0

Transition matrices

$\delta_0(a)$

1	2
0	1

$\delta_+(a)$

1	0
0	1

$\delta_+(b)$

1	0
0	1

Final distribution

ρ

0
1

- Let $w = aababba$.
- $\delta(w) = \delta_0(a)\ \delta_+(a)\ \delta_+(b)\ \delta_+(a)\ \delta_+(b)\ \delta_+(b)\ \delta_0(a)$.
- Accepting weight of $w = \lambda\delta(w)\rho$.

Motivation

- Decidability of equivalence of pPDA is an open problem [3].
- Equivalence of probabilistic OCA is also not known.
- Probabilistic ODCA is a class of probabilistic OCA for which equivalence is decidable.
- This is a strict super class of visibly probabilistic OCA.

Our equivalence results

- Equivalence of deterministic ODCA is in P.
  1. Deterministic OCAs are equivalent to deterministic ODCAs.
  2. Equivalence of deterministic OCA is in P [2].

Our equivalence results

- Equivalence of deterministic ODCA is in P.
  1. Deterministic OCAs are equivalent to deterministic ODCAs.
  2. Equivalence of deterministic OCA is in P [2].
- Equivalence of non-deterministic ODCA is in PSPACE.
  1. A non-deterministic ODCA is equivalent to an exponential sized deterministic ODCA.
  2. A PSPACE machine can do this.

Our equivalence results

- Equivalence of deterministic ODCA is in P.
  1. Deterministic OCAs are equivalent to deterministic ODCAs.
  2. Equivalence of deterministic OCA is in P [2].
- Equivalence of non-deterministic ODCA is in PSPACE.
  1. A non-deterministic ODCA is equivalent to an exponential sized deterministic ODCA.
  2. A PSPACE machine can do this.
- Equivalence of weighted ODCA is in P (main result).
  1. Equivalence of weighted finite automata is in P [6] (weights from a field).
  2. We reduce equivalence of weighted ODCA to that of weighted finite automata.

Reachability problem

Reachability

co-VS Reachability problem

INPUT:

- A weighted visibly OCA $\mathcal{A}$ over a field,
- an initial configuration $\mathtt{c}$,
- a vector space $\mathcal{V}$, and
- counter value m.

Reachability

co-VS Reachability problem

INPUT:

- A weighted visibly OCA $\mathcal{A}$ over a field,
- an initial configuration c,
- a vector space $\mathcal{V}$, and
- counter value m.

OUTPUT:

- *Yes*, if there exists a run $\mathsf{c} \xrightarrow{*} \overline{\mathcal{V}} \times \{m\}$ in $\mathcal{A}$, and
- *No*, otherwise.

Reachability

co-VS Reachability problem

INPUT:

- A weighted visibly OCA $\mathcal{A}$ over a field,
- an initial configuration c,
- a vector space $\mathcal{V}$, and
- counter value m.

OUTPUT:

- *Yes*, if there exists a run $\mathsf{c} \xrightarrow{*} \overline{\mathcal{V}} \times \{m\}$ in $\mathcal{A}$, and
- *No*, otherwise.

- $z \in \Sigma^*$ is a reachability witness for $(\mathsf{c}, \overline{\mathcal{V}}, m)$ if $\mathsf{c} \xrightarrow{z} \overline{\mathcal{V}} \times \{m\}$.

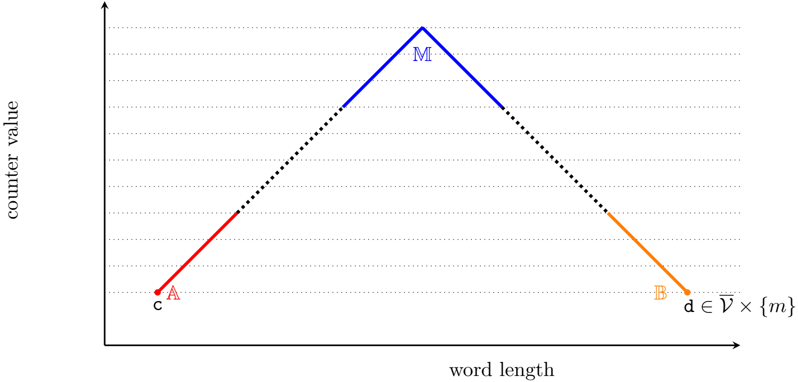
co-VS reachability

Theorem - co-VS reachability

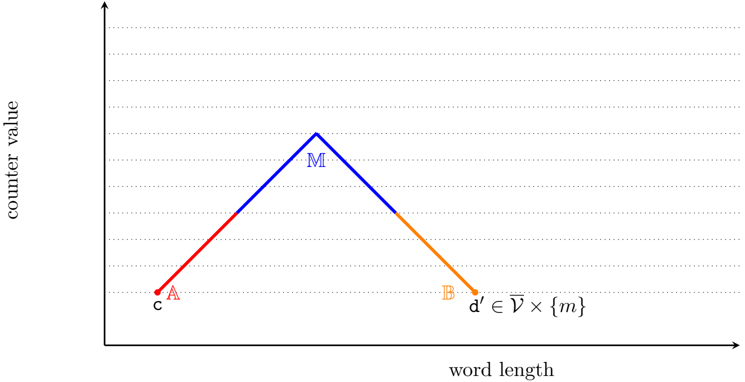
co-VS reachability is decidable in polynomial time.

- We prove this by showing a pseudo-pumping lemma.

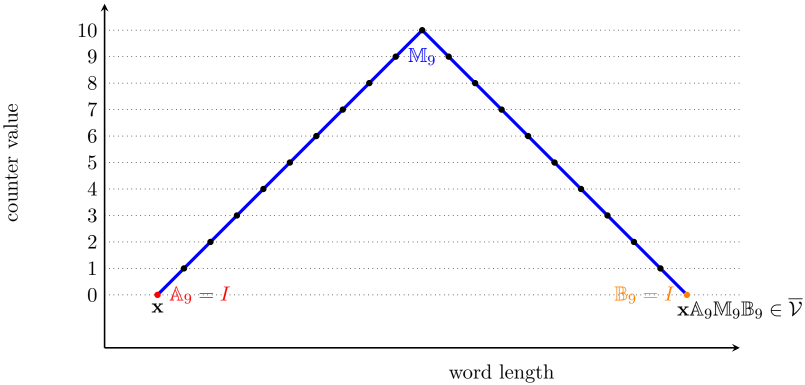
Pseudo-pumping lemma (pumping down)



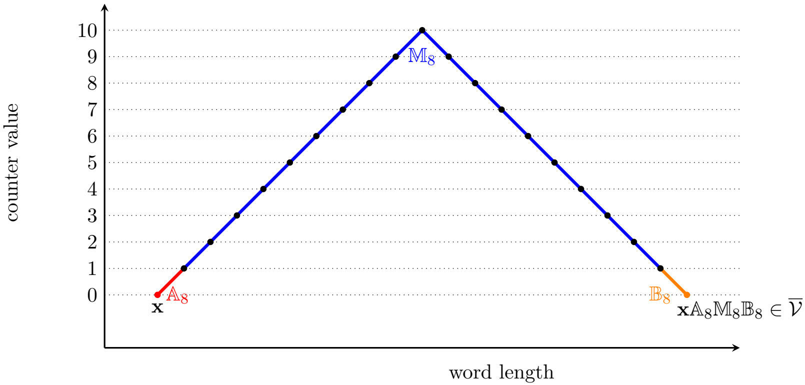
Pseudo-pumping lemma (pumping down)



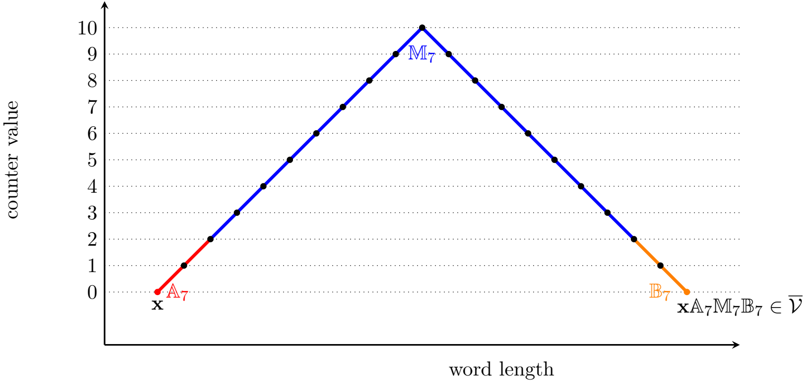
Pseudo-pumping lemma



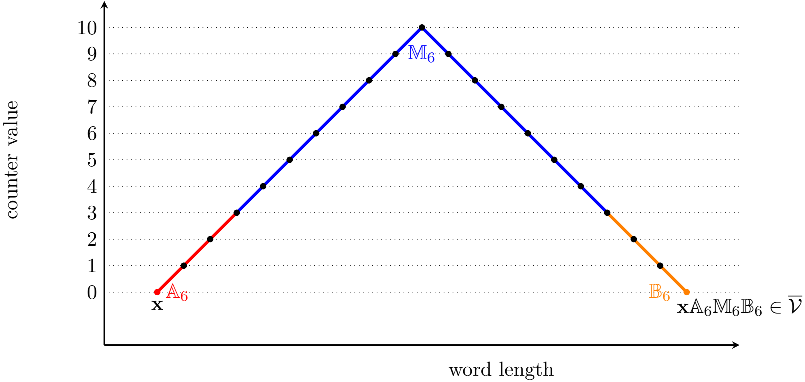
Pseudo-pumping lemma



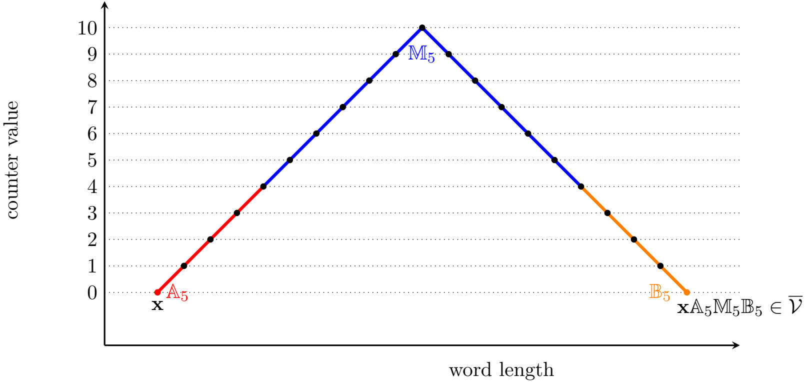
Pseudo-pumping lemma



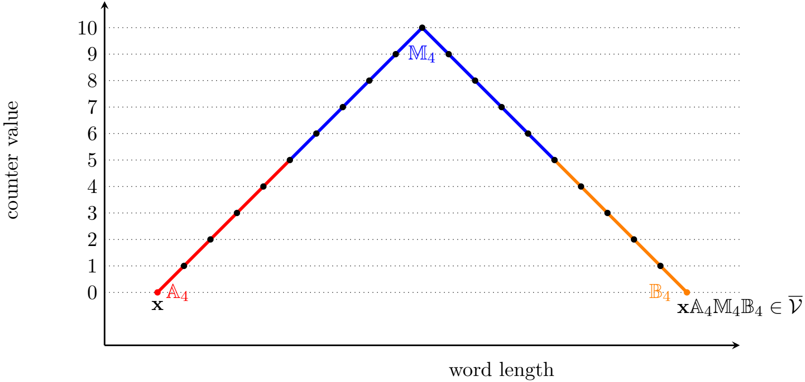
Pseudo-pumping lemma



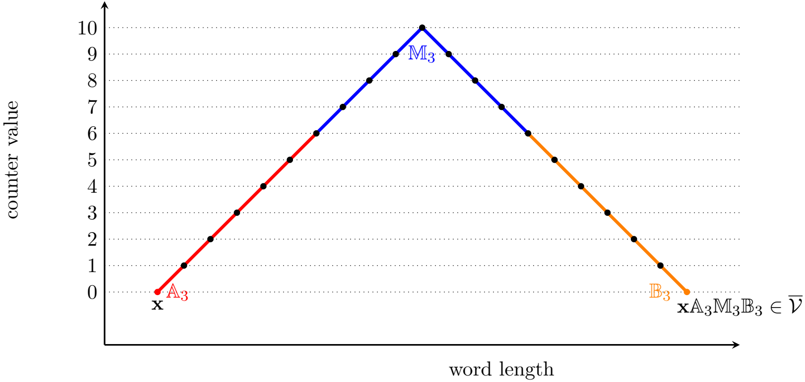
Pseudo-pumping lemma



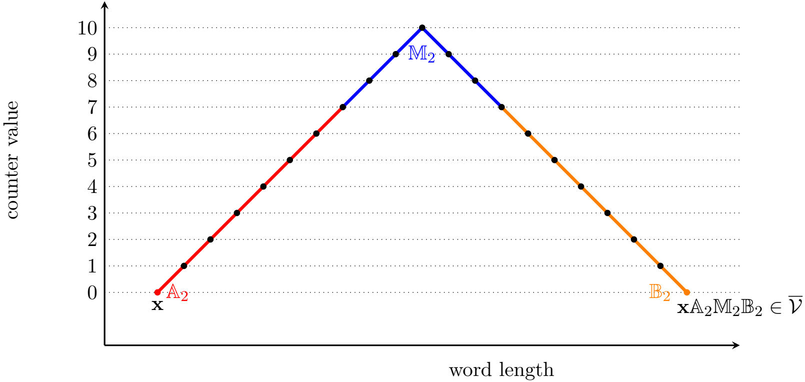
Pseudo-pumping lemma



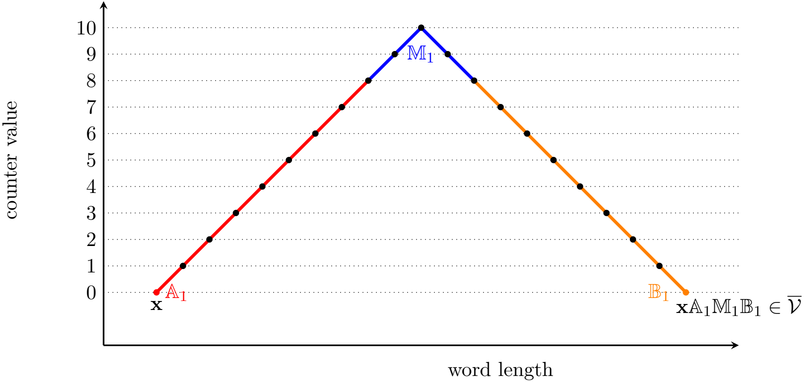
Pseudo-pumping lemma



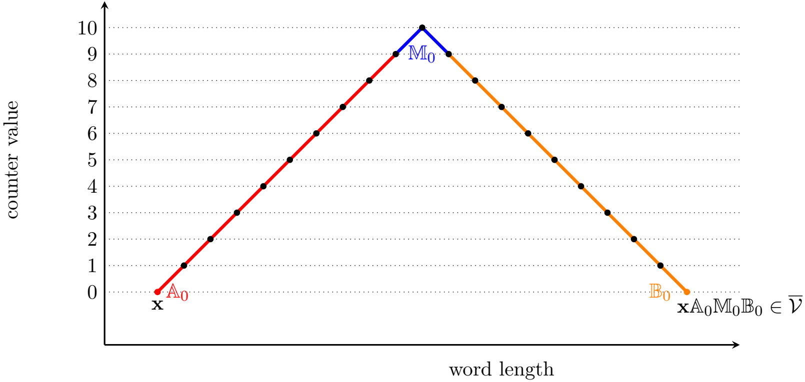
Pseudo-pumping lemma



Pseudo-pumping lemma

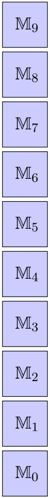


Pseudo-pumping lemma



Pseudo-pumping lemma

- Assume for all $i \in [0, 9]$, $\mathbb{M}_i$ is a 3×3 matrices.



Pseudo-pumping lemma

- Assume for all $i \in [0, 9]$, $\mathbb{M}_i$ is a 3×3 matrices.
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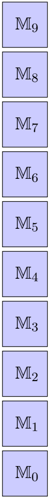

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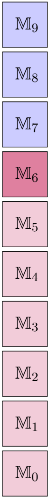

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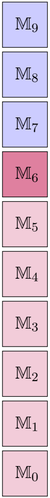
- $\mathbf{x}\mathbb{A}_j(\alpha_0\mathbb{M}_0 + \alpha_1\mathbb{M}_1 + \cdots + \alpha_{j-1}\mathbb{M}_{j-1})\mathbb{B}_j \in \overline{\mathcal{V}}$.



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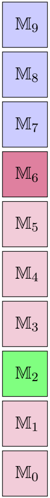
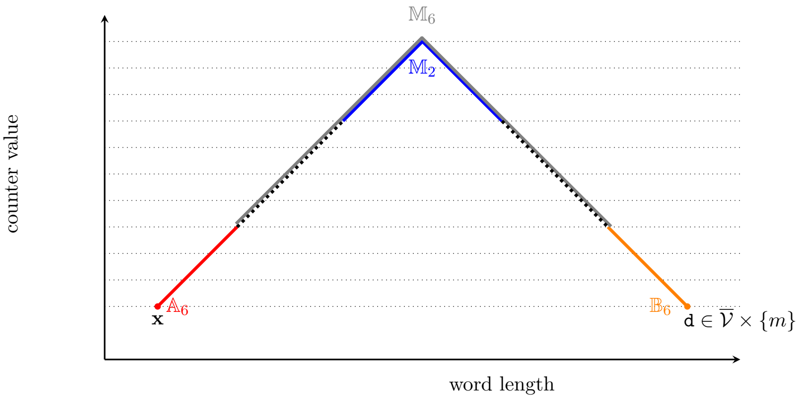
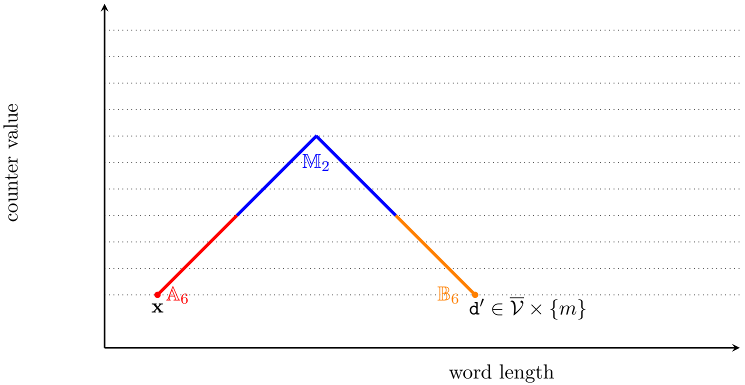


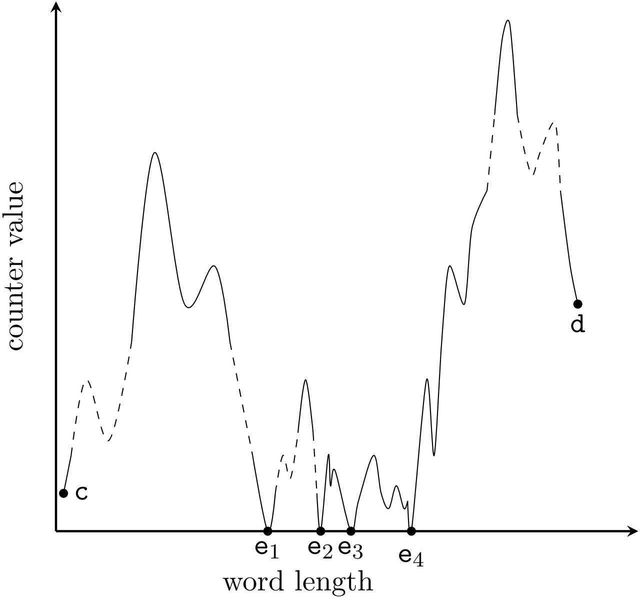
Pseudo-pumping lemma



Pseudo-pumping lemma



Multiple cuts



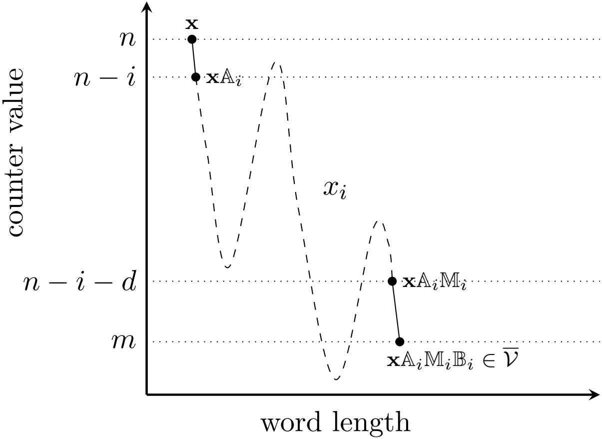
Special word Lemma

Lemma- Special word

The lexicographically minimal reachability witness z, if it exists, satisfies the following conditions :

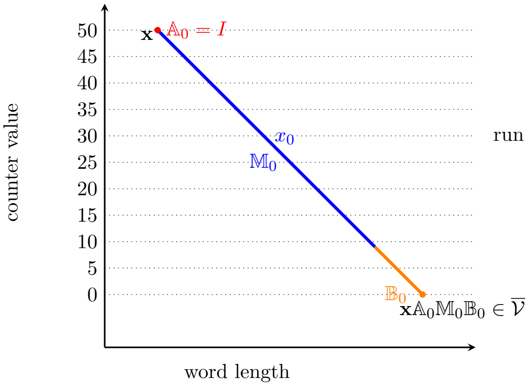
1. $z = uy_1^{r_1}w_1y_2^{r_2}w_2$ such that $|uy_1w_1y_2w_2|$ is polynomially bounded in input size, and
2. r_1, r_2 are polynomially bounded in input size.

Special word- Claim 1



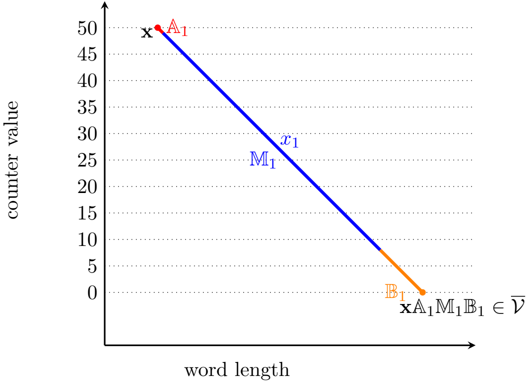
- We show that a factor repeats in the lexicographically minimal word if d is large enough.

Repeating subword



run of the lexicographically minimal witness z

Repeating subword



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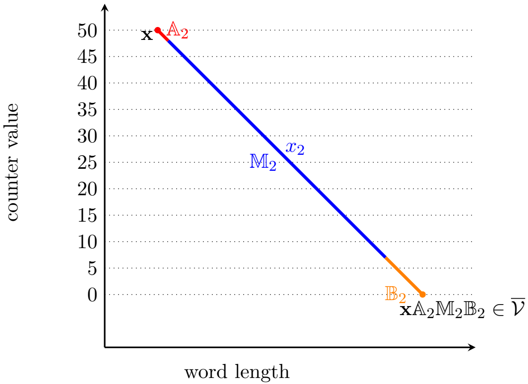
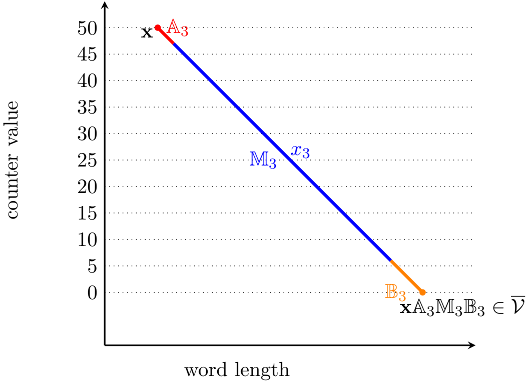


Repeating subword



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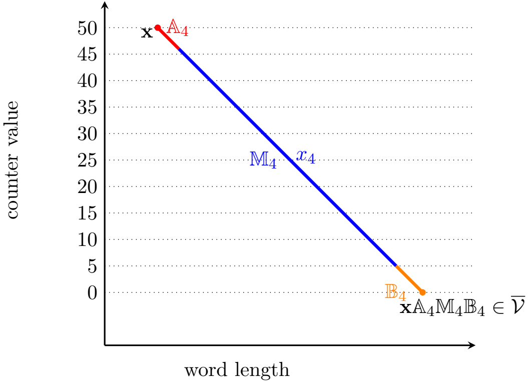

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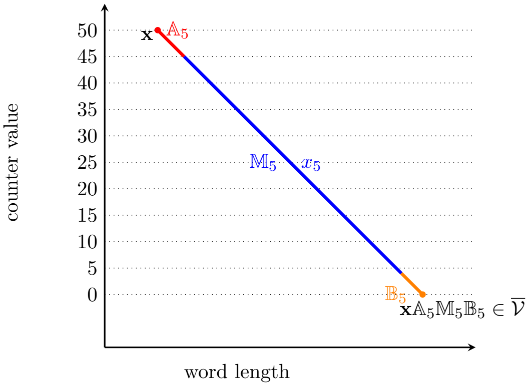

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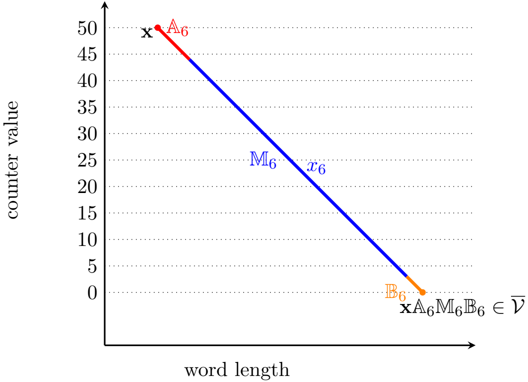

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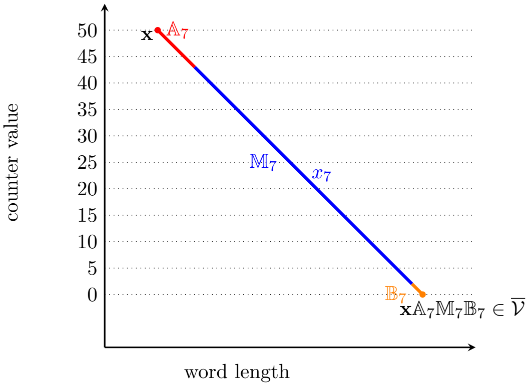

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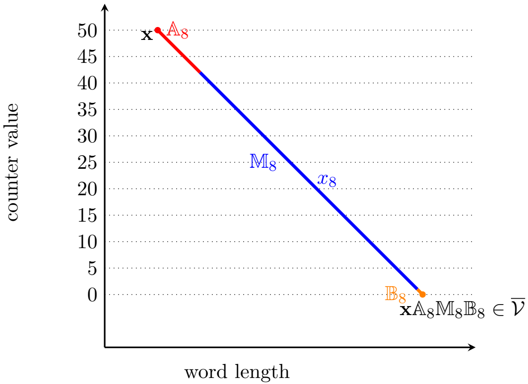

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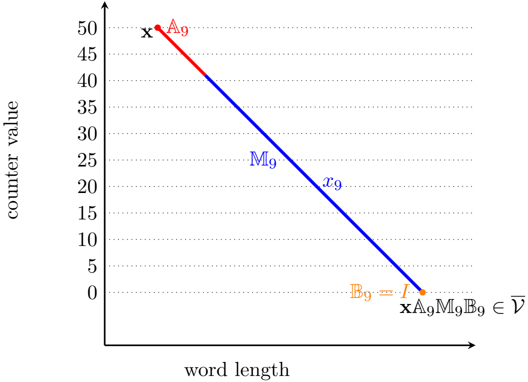

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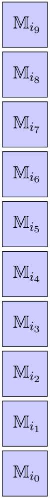
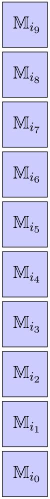


Special word- Claim 1

- Assume for all $i \in [0, 9]$, $\mathbb{M}_i$ is a 3×3 matrix.
- Arrange the words $x_0, x_1, \ldots, x_9$ in the length lex. ordering.



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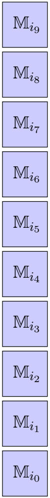

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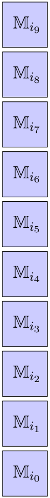

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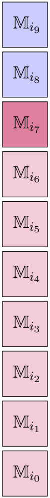

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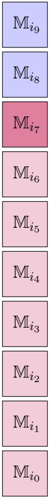

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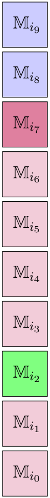
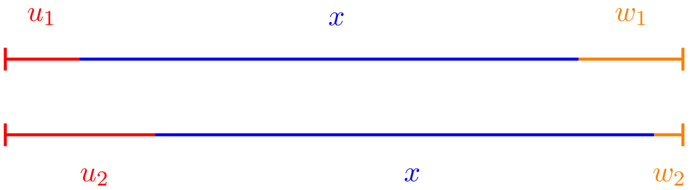


Special word- Claim 2



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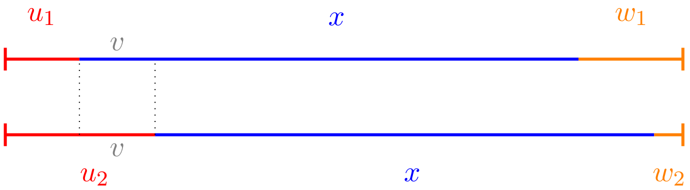

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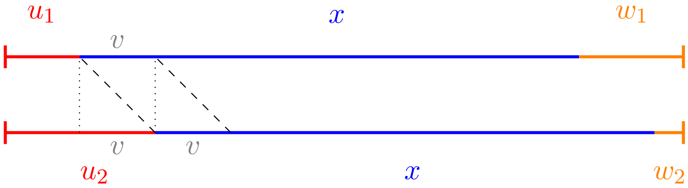

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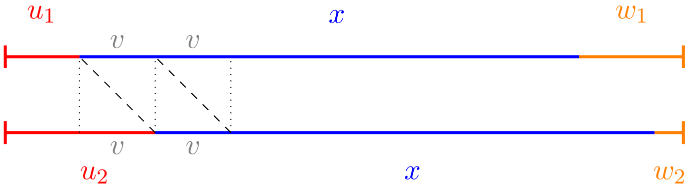

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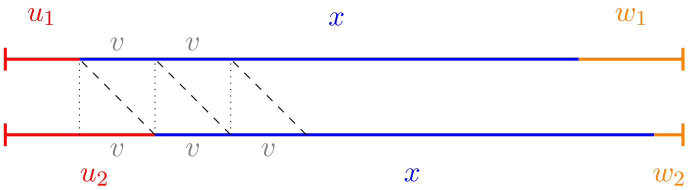

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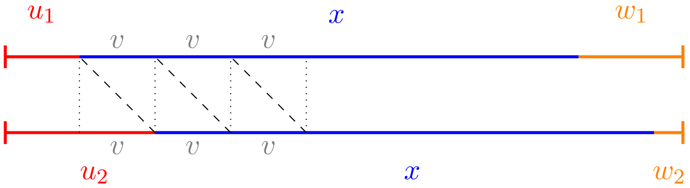

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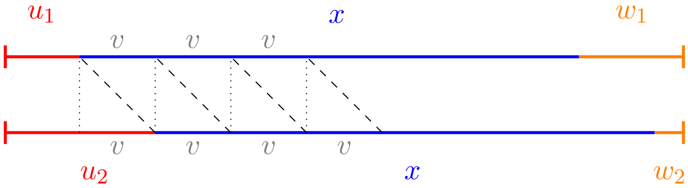

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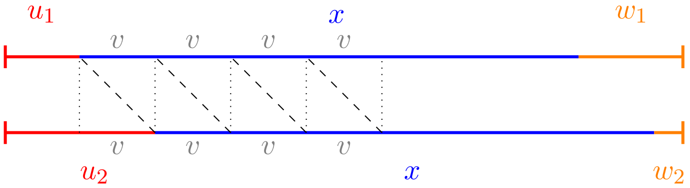

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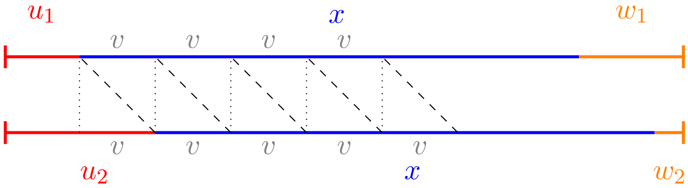

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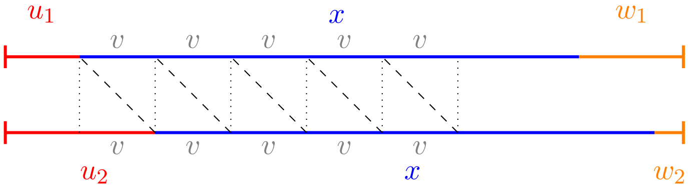

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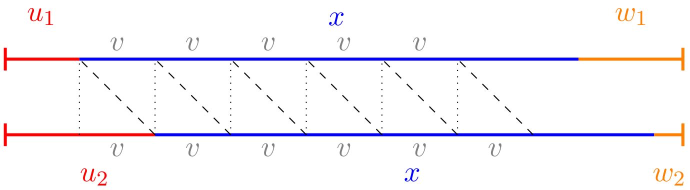

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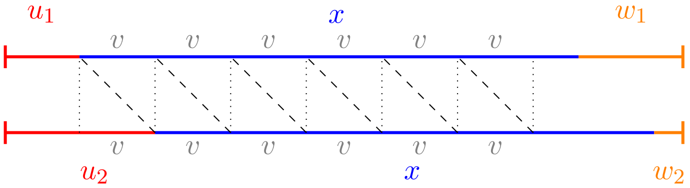

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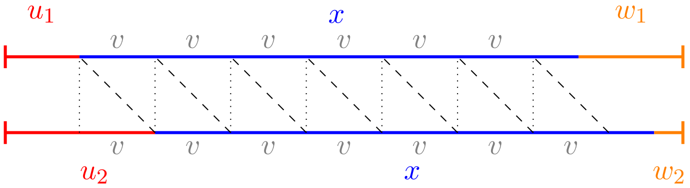
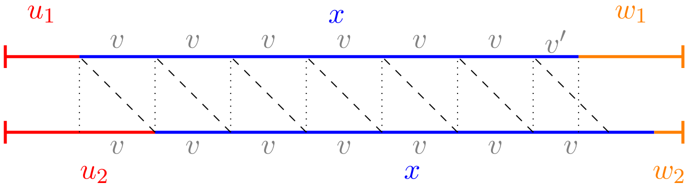


Special word- Claim 2



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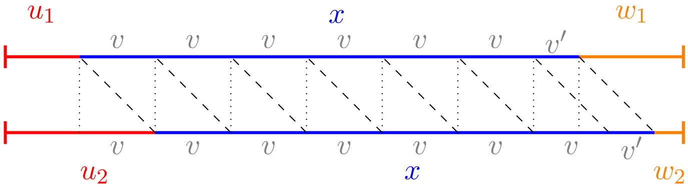
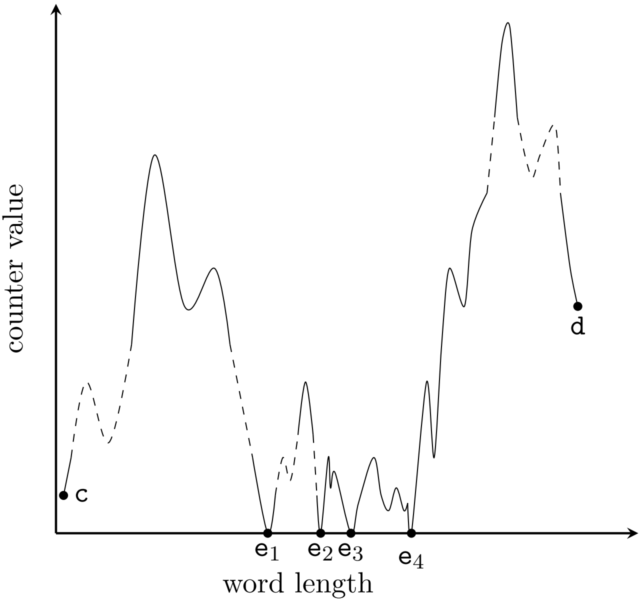


Multiple cuts



Equivalence

Equivalence

Lemma - Witness bound

If two weighted ODCAs $\mathcal{A}_1$ and $\mathcal{A}_2$ are not equivalent, then there exists a witness z such that the counter values encountered during the run of z are less than a polynomial in the input size.

Equivalence

Lemma - Witness bound

If two weighted ODCAs $\mathcal{A}_1$ and $\mathcal{A}_2$ are not equivalent, then there exists a witness z such that the counter values encountered during the run of z are less than a polynomial in the input size.

Theorem - Equivalence

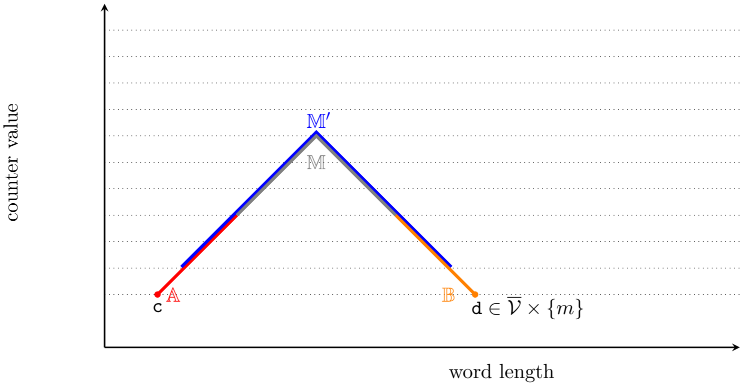
There is a polynomial time algorithm to check the equivalence of two weighted ODCAs (weights from a field).

Other results

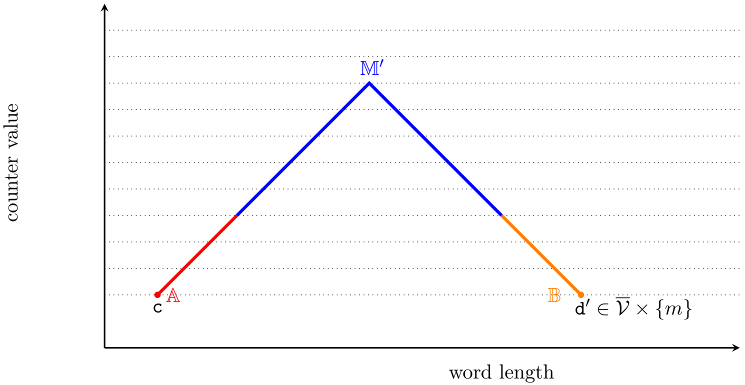
Theorem - Regularity

Given a weighted ODCA (weights from a field), determining whether there exists a weighted automata recognising the same function is in P.

Regularity - pumping up



Regularity - pumping up



Summary

- Equivalence of weighted ODCAs is in P.
- Regularity of weighted ODCAs is in P.

Existing Results

Machine	**Equivalence**	**Reference**
DFA	P (NL)	[4]
DOCA	P (NL-Complete)	[1]
NOCA	Undecidable	[7]
DPDA	Decidable	[5]
NPDA	Undecidable	[1]
PA	P	[6]
DWROCA	P	*Our result*
Weighted ODCA	P	*Our result*
Deterministic-stack PDA	*open*	-
pPDA	*open*	[3]

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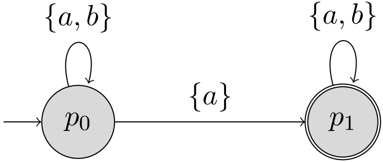
Thank You!

Appendix

Non-deterministic Finite Automata

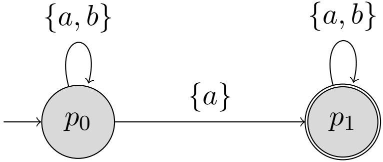
- Consider the following NFA.



- Language recognised?

Non-deterministic Finite Automata

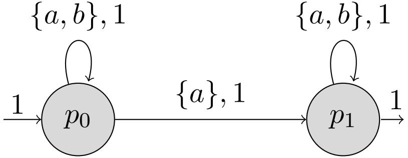
- Consider the following NFA.



- Language recognised?
  - Set of all words containing at least one 'a'.

Weighted Automata

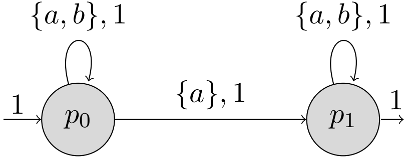
- A weighted automata (WA) can be viewed as an NFA with weights.
- In this talk, we assume that the weights always comes from a field.
- Consider the following WA.



- Function recognised?

Weighted Automata

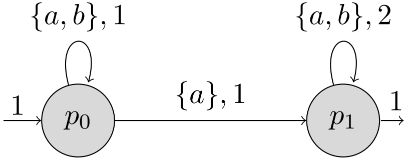
- A weighted automata (WA) can be viewed as an NFA with weights.
- In this talk, we assume that the weights always comes from a field.
- Consider the following WA.



- Function recognised?
  - Number of 'a's in the input.

Another example

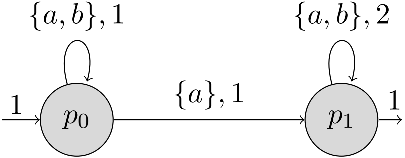
- Consider the following WA.



- Function recognised?

Another example

- Consider the following WA.



- Function recognised?
    - Decimal equivalent of binary number.

Equivalence WA(1)

- Given two weighted automata $\mathcal{A}$ and $\mathcal{B}$, the equivalence problem asks whether for all $w \in \Sigma^*$, $f_{\mathcal{A}}(w) = f_{\mathcal{B}}(w)$.

Theorem- Equivalence WA

The equivalence problem for weighted automata over a field is decidable in polynomial time.

- Given two weighted automata $\mathcal{A}$ and $\mathcal{B}$ over a field, we can construct a weighted automata $\mathcal{C}$ recognising the function $f_{\mathcal{A}} - f_{\mathcal{B}}$.
- To check equivalence of $\mathcal{A}$ and $\mathcal{B}$, it suffices to check the existence of a word which is accepted with non-zero weight by $\mathcal{C}$.

Equivalence(2)

- Let K be the number of states of $\mathcal{C}$ and η its final distribution.
- $\mathcal{V} = \{\mathbf{x} \in \mathcal{F}^{\mathsf{K}} \mid \mathbf{x}\eta = 0\}$ is a vector space.
- Given a vector space $\mathcal{U} \subseteq \mathcal{F}^{\mathsf{K}}$, we define $\overline{\mathcal{U}} = \mathcal{F}^{\mathsf{K}} \setminus \mathcal{U}$.
- The equivalence problem is to check the existence of a word w such that $\lambda\delta(w) \in \overline{\mathcal{V}}$.

co-VS reachability of weighted automata

co-VS reachability problem of WA

Input: Vector space $\mathcal{V}$, Weighted automaton $\mathcal{C}$.

Output: Yes, If there is a run of $\mathcal{C}$ that reaches a configuration in $\overline{\mathcal{V}}$.
No, otherwise.

Theorem- co-VS reachability

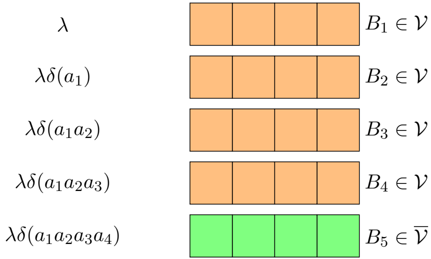
The co-VS reachability problem for weighted automata is decidable in polynomial time.

Lemma- minimal word

The length of the minimal run, if it exists, that reaches a configuration in $\overline{\mathcal{V}}$ is less than the number of states of $\mathcal{C}$.

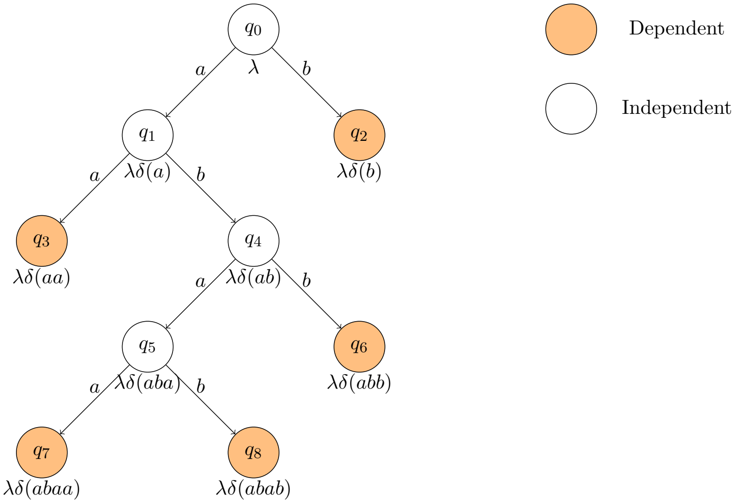
Minimal word

- Let the number of states of $\mathcal{C}$ be 4.
- Assume $w = a_1a_2a_3a_4$ is a minimal reachability witness.



- $B_5 = c_1B_1 + c_2B_2 + c_3B_3 + c_4B_4$ for some $c_1, c_2, c_3, c_4 \in \mathcal{F}$.
- There exists $i < 5$ such that $B_i \in \overline{\mathcal{V}}$. This is a contradiction.

Polynomial time algorithm



- Starting from q_0 construct a basis set by traversing the tree in a breadth-first manner and adding independent vectors to it.

Example ODCA (a)

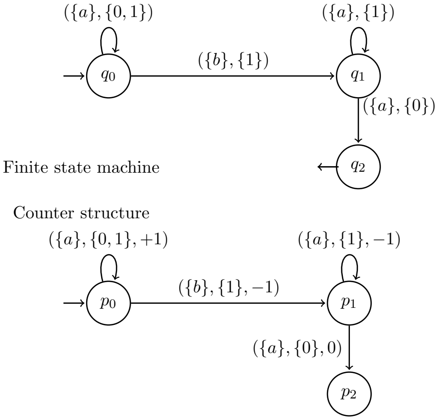
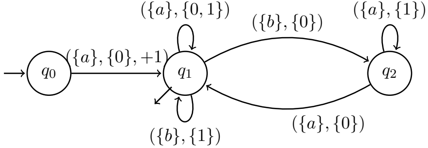


Figure: $\mathcal{L}_1 = \{a^n b a^n \mid n > 0\}$

Example ODCA (b)



Finite state machine

Counter structure

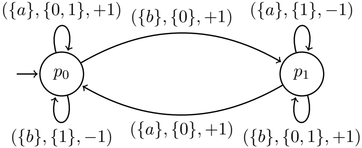


Figure: $\mathcal{L}_2 = \{(a+b)^* \mid \#\text{a's} > \#\text{b's}\}$

Example ODCA (c)

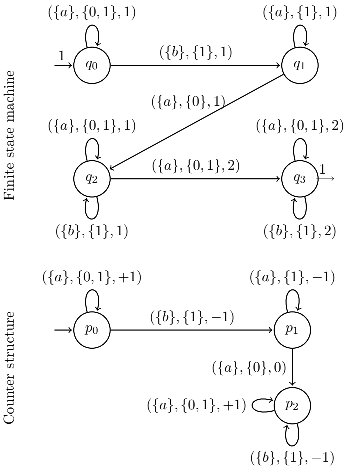
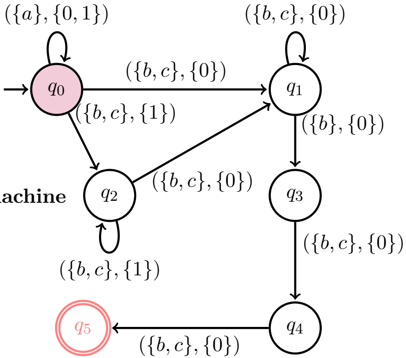


Figure: $f(w_1w_2)$ = decimal value of w_2's binary interpretation, if $w_1 \in \mathcal{L}_1$ and $\#a$'s $\geq \#b$'s for any prefix of w_2; 0 otherwise.

Example - One deterministic-counter Automata

$$L = \{a^m(b+c)^n b(b+c)^2 \mid n, m \in \mathbb{N} \text{ with } n > m\}.$$

Finite state machine



a	a	b	c	b	b	c	b

Input tape

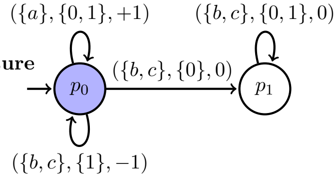
0

Counter

1	0	0	0	0	0

Initial vector

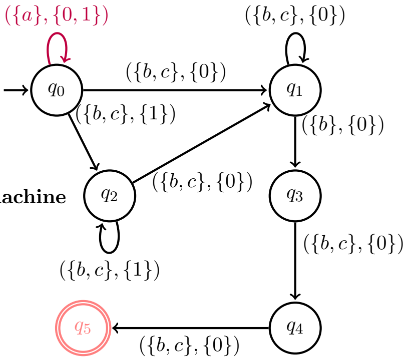
Counter structure



a	a	b	c	b	b	c	b

Input tape

Finite state machine

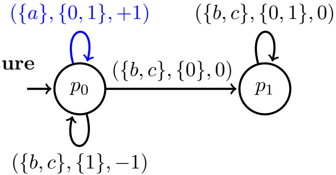


0

Counter

1	0	0	0	0	0

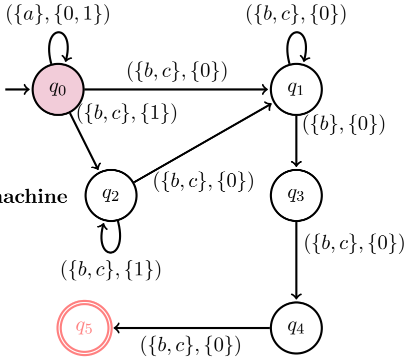
Counter structure



a	a	b	c	b	b	c	b

Input tape

Finite state machine

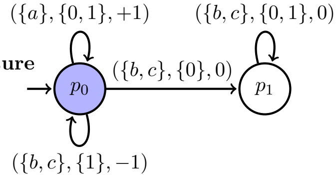


1

Counter

1	0	0	0	0	0

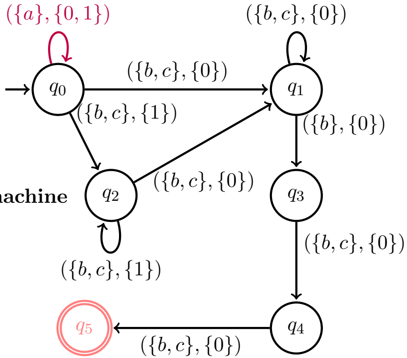
Counter structure



a	a	b	c	b	b	c	b

Input tape

Finite state machine

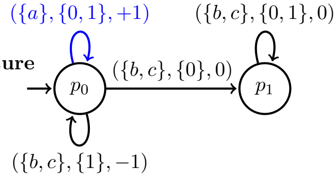


1

Counter

1	0	0	0	0	0

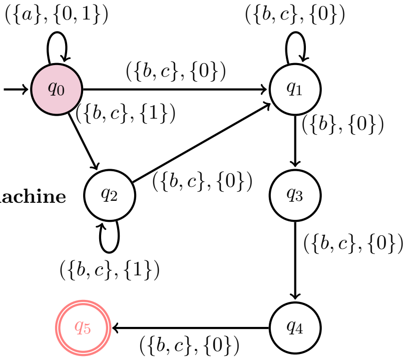
Counter structure



a	a	b	c	b	b	c	b

Input tape

Finite state machine

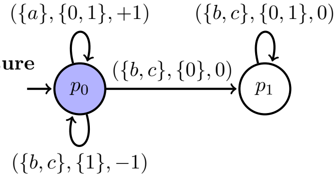


2

Counter

1	0	0	0	0	0

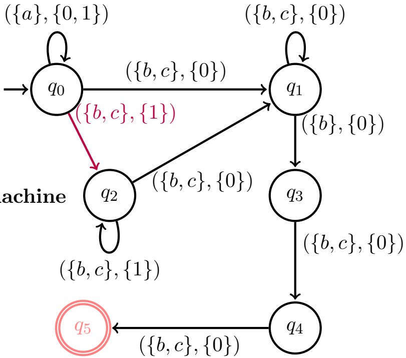
Counter structure



a	a	b	c	b	b	c	b

Input tape

Finite state machine

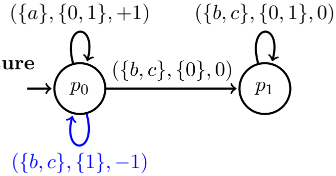


2

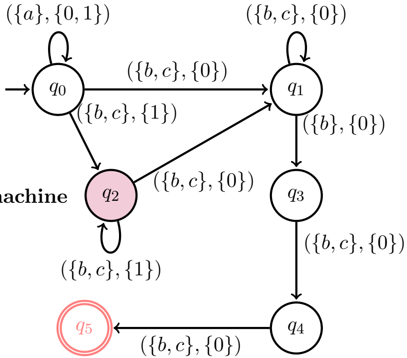
Counter

1	0	0	0	0	0

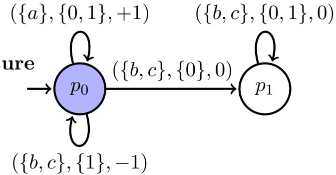
Counter structure



Finite state machine



Counter structure



a	a	b	c	b	b	c	b

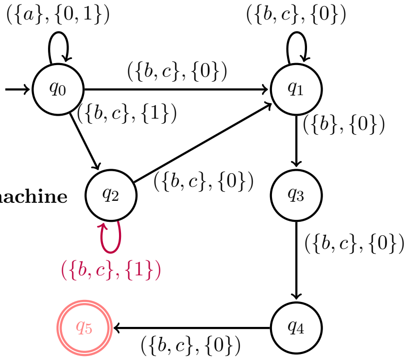
Input tape

1

Counter

0	0	1	0	0	0

Finite state machine



a	a	b	c	b	b	c	b

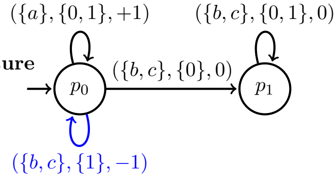
Input tape

1

Counter

0	0	1	0	0	0

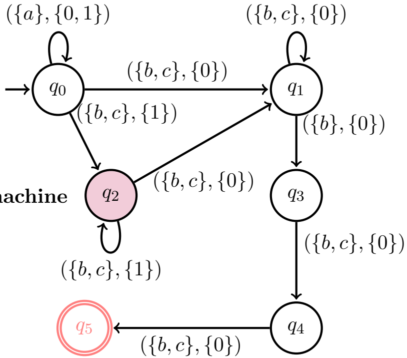
Counter structure



a	a	b	c	b	b	c	b

Input tape

Finite state machine

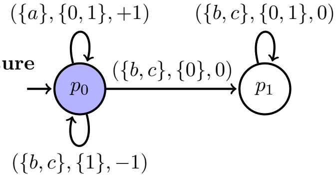


0

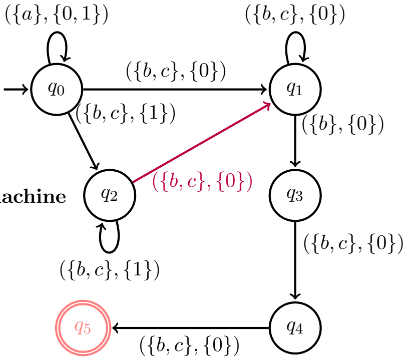
Counter

0	0	1	0	0	0

Counter structure



Finite state machine



a	a	b	c	b	b	c	b

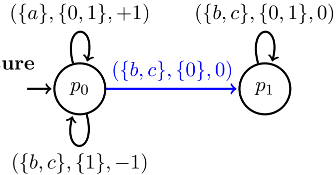
Input tape

0

Counter

0	0	1	0	0	0

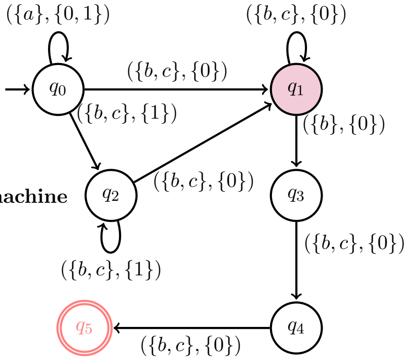
Counter structure



a	a	b	c	b	b	c	b

Input tape

Finite state machine

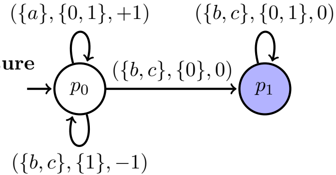


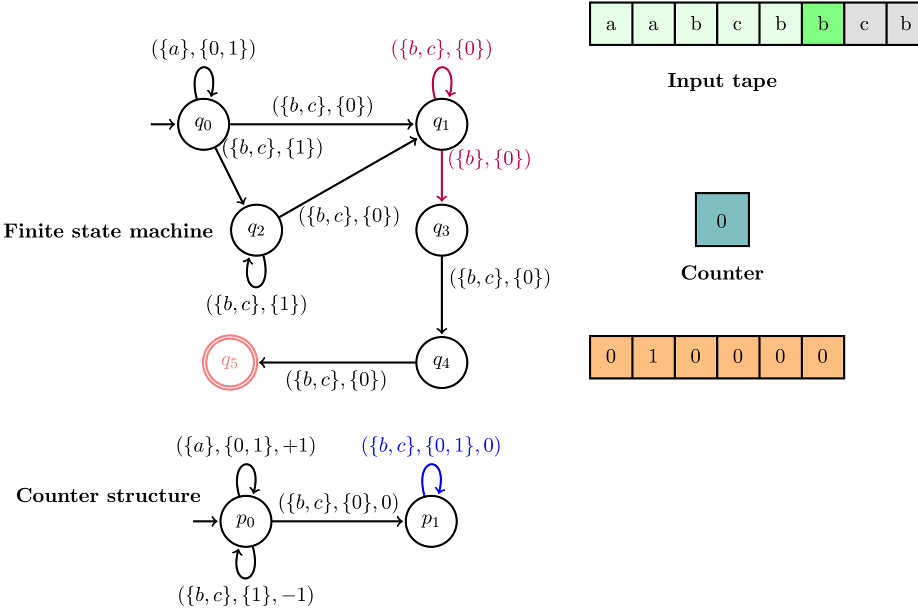
0

Counter

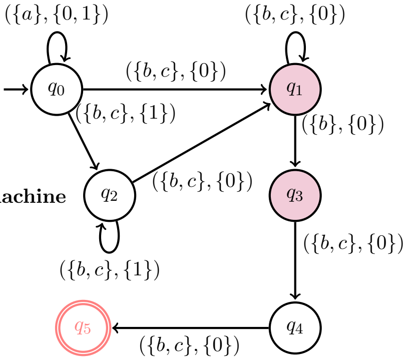
0	1	0	0	0	0

Counter structure




a a b c b b c b
Input tape
({a}, {0, 1})
({b, c}, {0})
q0
q1
({b, c}, {1})
({b}, {0})
Finite state machine
q2
({b, c}, {0})
q3
({b, c}, {0})
({b, c}, {1})
q5
q4
({b, c}, {0})
0
Counter
0 1 0 0 0 0
({a}, {0, 1}, +1)
({b, c}, {0, 1}, 0)
Counter structure
p0
({b, c}, {0}, 0)
p1
({b, c}, {1}, −1)

Finite state machine



a	a	b	c	b	b	c	b

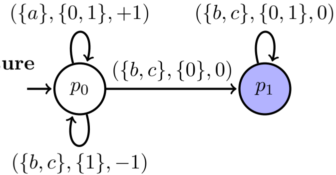
Input tape

0

Counter

0	1	0	1	0	0

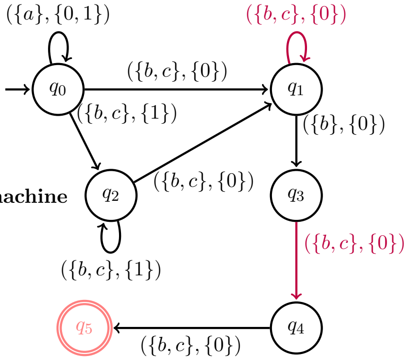
Counter structure



a	a	b	c	b	b	c	b

Input tape

Finite state machine

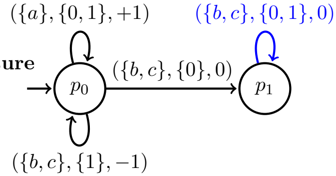


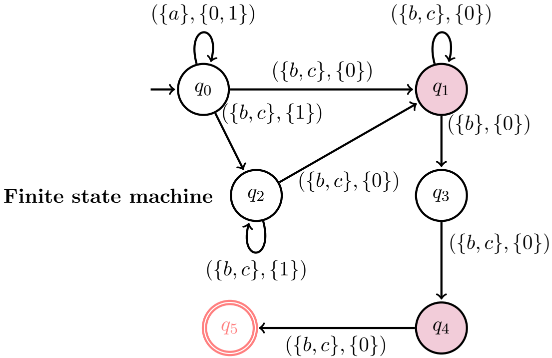
0

Counter

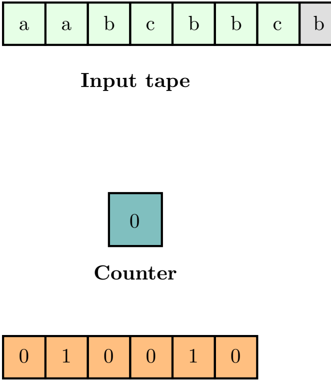
0	1	0	1	0	0

Counter structure

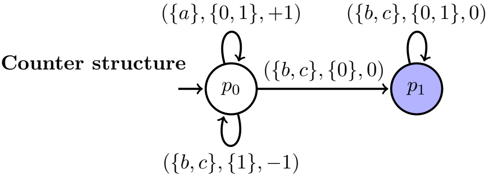




Finite state machine
q_0
q_1
q_2
q_3
q_4
q_5
$(\{a\},\{0,1\})$
$(\{b,c\},\{0\})$
$(\{b,c\},\{0\})$
$(\{b,c\},\{1\})$
$(\{b\},\{0\})$
$(\{b,c\},\{0\})$
$(\{b,c\},\{1\})$
$(\{b,c\},\{0\})$
$(\{b,c\},\{0\})$

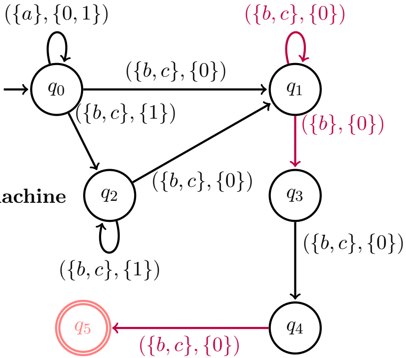


a a b c b b c b
Input tape
0
Counter
0 1 0 0 1 0



Counter structure
p_0
p_1
$(\{a\},\{0,1\},+1)$
$(\{b,c\},\{0,1\},0)$
$(\{b,c\},\{0\},0)$
$(\{b,c\},\{1\},-1)$

Finite state machine



a	a	b	c	b	b	c	b

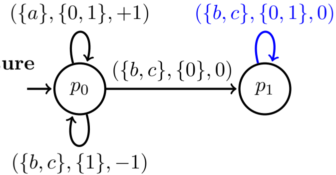
Input tape

0

Counter

0	1	0	0	1	0

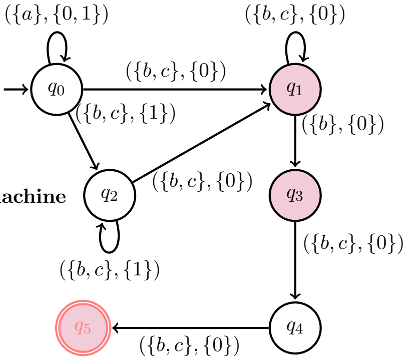
Counter structure



a	a	b	c	b	b	c	b

Input tape

Finite state machine

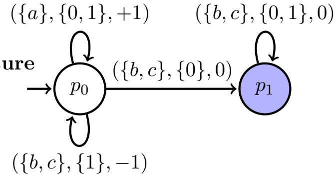


0

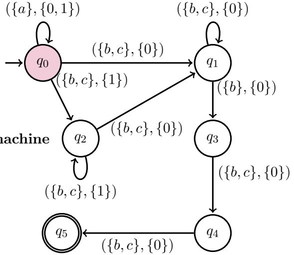
Counter

0	1	0	1	0	1

Counter structure

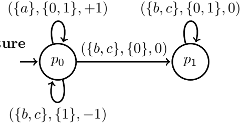


Finite state machine



x

Counter structure



0	0	1	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b\}, \{1\})$

0	1	0	0	0	0
0	1	0	1	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

$(\{b\}, \{0\})$

$(\{a\},\{0,1\})$

1	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b,c\},\{1\})$

0	0	1	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b\},\{0\})$

0	1	0	0	0	0
0	1	0	1	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

$(\{c\},\{0\})$

0	1	0	0	0	0
0	1	0	0	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

.

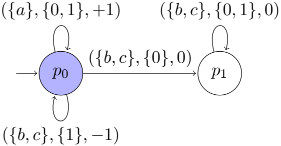
Transition matrices

a	c	c	b	c	b

Input tape

1	0	0	0	0	0

Initial vector



Counter structure

0

Counter

$(\{a\}, \{0,1\})$

1	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b,c\}, \{1\})$

0	0	1	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b\}, \{0\})$

0	1	0	0	0	0
0	1	0	1	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

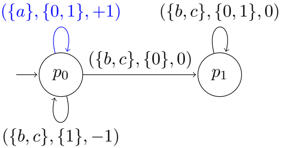
$(\{c\}, \{0\})$

0	1	0	0	0	0
0	1	0	0	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

.

a	c	c	b	c	b

1	0	0	0	0	0



0

Counter

$(\{a\},\{0,1\})$

1	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b,c\},\{1\})$

0	0	1	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b\},\{0\})$

0	1	0	0	0	0
0	1	0	1	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

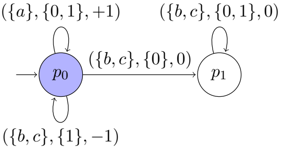
$(\{c\},\{0\})$

0	1	0	0	0	0
0	1	0	0	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

.

a	c	c	b	c	b

1	0	0	0	0	0



1

Counter

$(\{a\},\{0,1\})$

1	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b,c\},\{1\})$

0	0	1	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b\},\{0\})$

0	1	0	0	0	0
0	1	0	1	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

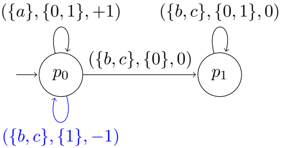
$(\{c\},\{0\})$

0	1	0	0	0	0
0	1	0	0	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

.

a	c	c	b	c	b

1	0	0	0	0	0



1

Counter

$(\{a\},\{0,1\})$

1	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b,c\},\{1\})$

0	0	1	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b\},\{0\})$

0	1	0	0	0	0
0	1	0	1	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

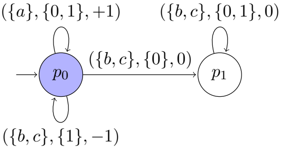
$(\{c\},\{0\})$

0	1	0	0	0	0
0	1	0	0	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

.

a	c	c	b	c	b

0	0	1	0	0	0



0

Counter

$(\{a\}, \{0,1\})$

1	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b,c\}, \{1\})$

0	0	1	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b\}, \{0\})$

0	1	0	0	0	0
0	1	0	1	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

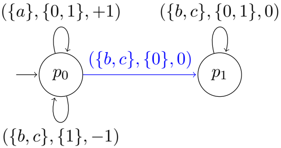
$(\{c\}, \{0\})$

0	1	0	0	0	0
0	1	0	0	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

.

a	c	c	b	c	b

0	0	1	0	0	0



0

Counter

$(\{a\},\{0,1\})$

1	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b,c\},\{1\})$

0	0	1	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b\},\{0\})$

0	1	0	0	0	0
0	1	0	1	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

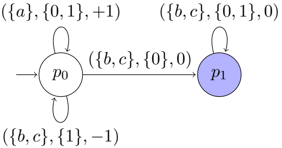
$(\{c\},\{0\})$

0	1	0	0	0	0
0	1	0	0	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

.

a	c	c	b	c	b

0	1	0	0	0	0



0

Counter

$(\{a\},\{0,1\})$

1	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b,c\},\{1\})$

0	0	1	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b\},\{0\})$

0	1	0	0	0	0
0	1	0	1	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

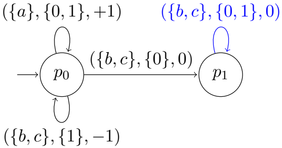
$(\{c\},\{0\})$

0	1	0	0	0	0
0	1	0	0	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

.

a	c	c	b	c	b

0	1	0	0	0	0



0

Counter

$(\{a\}, \{0,1\})$

1	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b, c\}, \{1\})$

0	0	1	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b\}, \{0\})$

0	1	0	0	0	0
0	1	0	1	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

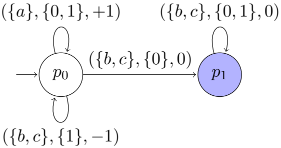
$(\{c\}, \{0\})$

0	1	0	0	0	0
0	1	0	0	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

.

a	c	c	b	c	b

0	1	0	1	0	0



0

Counter

$(\{a\},\{0,1\})$

1	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b,c\},\{1\})$

0	0	1	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b\},\{0\})$

0	1	0	0	0	0
0	1	0	1	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

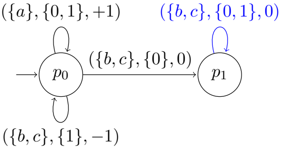
$(\{c\},\{0\})$

0	1	0	0	0	0
0	1	0	0	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

.

a	c	c	b	c	b

0	1	0	1	0	0



0

Counter

$(\{a\},\{0,1\})$

1	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b,c\},\{1\})$

0	0	1	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b\},\{0\})$

0	1	0	0	0	0
0	1	0	1	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

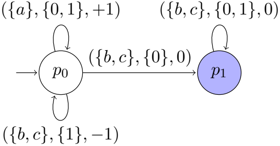
$(\{c\},\{0\})$

0	1	0	0	0	0
0	1	0	0	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

.

a	c	c	b	c	b

0	1	0	0	1	0



0

Counter

$(\{a\},\{0,1\})$

1	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b,c\},\{1\})$

0	0	1	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b\},\{0\})$

0	1	0	0	0	0
0	1	0	1	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

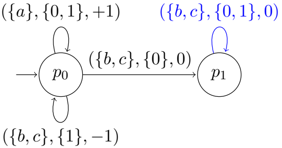
$(\{c\},\{0\})$

0	1	0	0	0	0
0	1	0	0	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

.

a	c	c	b	c	b

0	1	0	0	1	0



0

Counter

$(\{a\}, \{0, 1\})$

1	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b, c\}, \{1\})$

0	0	1	0	0	0
0	0	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$(\{b\}, \{0\})$

0	1	0	0	0	0
0	1	0	1	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

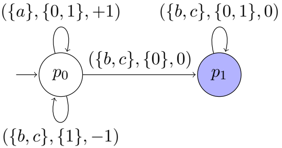
$(\{c\}, \{0\})$

0	1	0	0	0	0
0	1	0	0	0	0
0	1	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0

.

a	c	c	b	c	b

0	1	0	1	0	1



$\boxed{0}$

Counter

Pseudo-pumping lemma (pumping down)

Pseudo-pumping lemma

If z is a reachability witness for $(\mathsf{c}, \overline{\mathcal{V}}, m)$ and the maximum counter value encountered during the run of z is not polynomially bounded in the input size then,

1. there exists a subword z_{sub} of z, such that $\mathsf{c} \xrightarrow{z_{sub}} \overline{\mathcal{V}} \times \{m\}$, and the maximum counter value encountered during this run is less than the maximum counter value encountered during the run of z.

Pseudo-pumping lemma - Figure